A THEORETICAL REVIEW OF FD STATISTICS IN CLASSICAL LIMIT: EQUIVALENCE OF FD AND MB STATISTICS

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Abstract: In this brief communication, a systematic analysis on the FD distribution law is presented. It has been elucidated that in classical limit of high temperature or low density, FD statistics reduces to MB statistics. An empirical relation underlying the transition from Quantum to Classical statistics is reported in the present article. The equivalence in the nature of MB and FD distribution functions above certain critical energy value is depicted graphically. The fact has been established in the context of plasma electrons admissible in the near sheath region of laboratory dusty plasma set up. Coulomb coupling parameter for plasma electrons in dusty plasma environment is estimated to be much smaller than unity. The consideration of Maxwellian electrons has correctly accounted for the physical properties of laboratory dusty plasma system. An analytical justification of classical behaviour in fermions is presented in this article.

Keywords: FD statistics; MB statistics; dusty plasma; fermions and Maxwell Boltzmann particles

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1 Introduction

Statistical Mechanics is a branch of Physics that comprehends the behaviour of many particle systems found in nature [1]. These systems are admissible in terrestrial as well as extraterrestrial world. Statistical Physics bridges the gap between macroscopic properties of a physical system and the dynamical laws governing the evolution of microscopic constituents in a physical system [1]. A system of gas molecules obey Maxwell Boltzmann statistics (MB) which is exclusively applicable for systems that behave classically [2]. The gas molecules stay sufficiently apart thereby manifesting a character of distinguishability [1]. The behaviour of physical systems comprising of bosons (example-photons) is governed by Bose-Einstein (BE) statistics. BE statistics govern the physical behaviour of systems comprising of particles with integral spin. On the other hand, the statistics of systems comprising of half integral spin particles called fermions (example-electrons) is known as the Fermi-Dirac (FD) statistics. BE and FD statistics can explain the nature of systems that behave Quantum mechanically [3]. Thus, the two statistics (BE and FD) are collectively referred to as Quantum statistics. Quantum Statistics is operational only in the limit of low temperature and high density. In this thermodynamic limit, the system is strongly degenerate and Quantum mechanical effects are predominant [2]. The particles retain indistinguishability. However, the system of Bosons or Fermions can behave classically when the temperature of the system is reasonably high and the density is low [3]. Thus, Quantum statistics can get reduced to Classical in high temperature condition or low density regime. However in such continuous limit of transition the particles continue to be indistinguishable [4]. The reduction of Quantum statistics to classical is discussed by Barletti et. al [5] with the derivation of Quantum fluid equations for particles obeying FD and BE statistics. The work analytically shows that a semiclassical expansion of Quantum fluid equations to $O(\hbar^2)$ leads to classical fluid equations [5]. In many particle systems like dusty plasma, the plasma electrons behave classically in certain parameter regime. The electrons that interact with the heavy dust particles levitated in the near sheath region of dusty plasma set up follows MB distribution at high temperature limit. The theoretical work by Bezbaruah et. al [6] has established the presence of inertia less electrons following MB distribution in typical laboratory conditions. It is accounted that the electron susceptibility term in Dielectric Response Function of dusty plasma is devoid of any dependence on model parameters such as external field strength or inter particle collision frequency. In the context of experimental result, Allen and Chen [7,8] have explained the relevance of Boltzmannian electron in dusty plasma set up. Allen discussed the fact that the electron density obeys MB distribution as one approaches the plasma boundary where the electric term in Lorentz force expression dominates over magnetic term and control the dynamics [7,9]. It is also observed that the MB distribution law obeys an exponential variation with energy and it reveals that the occupation index asymptotically approaches zero as energy value goes to infinity. The distribution law governing Quantum statistics for Bosons and Fermions also have exponential dependence on energy value. However, the distribution law for FD statistics directly refer to probability that a state is occupied by a particle and that for BE and MB statistics give the most probable number per quantum state [10]. The central idea of Statistical Mechanics attempts to evaluate the number of microstates corresponding to a given macrostate describing the system. It can account for the thermodynamic properties of the system under consideration. MB, BE and FD statistics have distinct expressions for computing the number of microstates. The most probable state describes the equilibrium condition in a physical system and the macrostate with highest number of microstates is recognized as the most probable state [3].

The many body systems in the Universe are subjected to the laws of Classical or Quantum statistics. The Distribution law dictates the behaviour of a physical system. Thus, it is of great importance to unravel the physics of Classical and Quantum statistics and to investigate the parameter space in which the two statistics essentially unites.

2 Theoretical Framework

2.1 Description of the Fermi Dirac (FD) system

Let us consider a system of fermions (say electrons) comprising a FD system. 'N' be the total number of fermions in the many body system. The electrons in the system are going to be distributed in some energy levels that constitute the system. One may draw an analogy of the distribution of particles with that of students in desk-bench comprising a classroom. We may suppose that 'ε_1', 'ε_2', 'ε_3'……'ε_i'…. be the energy levels associated with respective statistical weight factors 'g_1', 'g_2', 'g_3…….'g_i'…..in a system of particles. It is considered that 'n_1' particles will occupy energy level 'ε_1', 'n_2' particles will occupy level 'ε_2' and so on.

Fermions are essentially indistinguishable particles that obey Pauli's Exclusion Principle (PEP). The PEP is connected to anti symmetric nature of wave functions associated with these particles. The matter waves associated with the particles are described by wave functions. The overlapping of these wave functions associated with two fermions tends to cancel each other thereby prohibiting the multiple occupancy of quantum states in FD system.

The number of microstates accessible for the FD system describes the possible number of ways in which the fermions can get arranged in the energy levels and hence it measures the thermodynamic probability. Following the conditions operative [1,2] in the FD system the thermodynamic probability can be stated as-

$$\omega_{FD} = \prod_{i=1}^{k} \frac{g_i!}{n_i!(g_i - n_i)!} \tag{1}$$

where the symbols have their usual meaning.

The above expression for thermodynamic probability can be used to estimate the FD distribution law:

$$f_{FD} = \frac{n_i}{g_i} = \frac{1}{e^{\alpha} e^{\frac{\varepsilon_i}{KT}} + 1} \tag{2}$$

Here, $\alpha = \frac{-\mu}{KT}$ is the Lagrangian multiplier, defined as the ratio of chemical potential to thermal energy. Equation (2) describes the occupation index of FD system. The factor "1" in the denominator ensures that the value of distribution function f_{FD} cannot exceed unity. It is useful to account for the probability that a quantum state will get occupied by a fermion.

2.2 Reduction of FD statistics to Classical statistics-

Under the limit of high temperature and low density, the behaviour of FD system undergoes a drastic transition. The behaviour of fermions reveals a significant change in the dynamics of the system.

The mean thermal wavelength associated with the particles is defined as [3]-

$$\lambda_T = \frac{h}{\sqrt{2\pi m K T}} \tag{3}$$

At high temperature, the mean thermal wavelength of matter waves associated with the particles (fermions) decreases significantly in comparison to the average inter particle distance r_{av} ($r_{av} \propto n^{-\frac{1}{3}}$, where n is number density). It thereby supresses the wave character associated with the particles. Thus, we approach a limiting condition where $r_{av} \gg \lambda_T$ [2] , the particles are less affected by the wave characteristics and these particles behave independently just like molecules of ideal gas.

In classical systems like the molecules in a gas, we may consider infinitely small size of the cells constituting the phase space. Thus, the statistical weight factor is large enough in comparison to the total number of particles going to be accommodated in a particular energy level. The probability that two particles occupy same quantum state is reasonably small despite the fact that there is no restriction in occupancy imposed for Classical systems.

2.2.1 Equivalence of FD and MB statistics

The number of microstates or the thermodynamic probability of Classical systems obeying Maxwell Boltzmann (MB) statistics is [2]-

$$\omega_{MB} = N! \prod_{i=1}^{k} \frac{g_i^{n_i}}{n_i!} \tag{4}$$

Incorporating Gibb's correction that accounts for removal of the consideration that classical particles are distinguishable, expression (4) reduces to the form-

$$\omega_{MB} = \prod_{i=1}^{k} \frac{g_i^{n_i}}{n_i!} \tag{5}$$

Equation (1) and (5) essentially give the same number of microstates under the condition of high temperature and low density. The MB thermodynamic probability can be redefined considering that the individual particles are occupying distinct cells in phase space under a certain thermodynamic condition. The number of ways for arranging 1st particle from a group of 'n_i' in the energy level 'ε_i'is 'g_i', the second particle will preferably choose any cells from a total of '$g_i - 1$', the third will choose any cell from a total of '$g_i - 2$' and so on. Thus, the expression (5) eventually reduces to expression (1).

The MB distribution law can be stated as [1]-

$$f_{MB} = \frac{n_i}{g_i} = \frac{1}{e^{\alpha} e^{\frac{\varepsilon_i}{KT}}} \tag{6}$$

Here, the symbols have their usual meaning. In the FD distribution law i.e equation (2), the unity factor in the denominator can be neglected when-

$$e^{\frac{\varepsilon-\mu}{KT}} \gg 1 \quad (7)$$

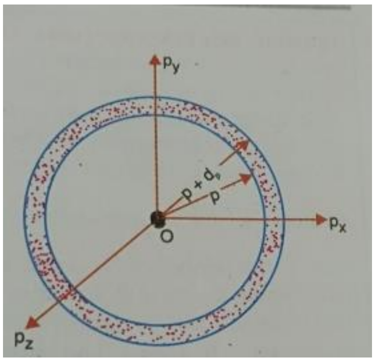
If we approximate that the exponential factor in the LHS of inequality relation (7) is around 10^2 or more than the requirement for reduction of FD distribution law (equation (2)) to MB distribution law (equation (6)) may be fulfilled.

Thus,

$$e^{\frac{\varepsilon-\mu}{KT}} \geq 10^2 \quad (8)$$

$$\text{Or, } \frac{\varepsilon-\mu}{KT} \geq 2\log_e^{10} \quad 9)$$

Inequality relation (9) depicts an empirical relation which can establish equivalence between MB and FD statistics. The equivalence is usually established in a limit where the occupation index lies below unity. The variation of occupation index with energy follows exponential decay.



The statistical weight factor associated with MB and FD system can be computed in phase space as-

$$g(p)dp = \frac{4\pi p^2 V\gamma dp}{h^3} \quad (10)$$

where V is the spatial volume and γ is the spin degeneracy term. The statistical weight factor is a measure of the number of quantum states available in the momentum range 'p' to 'p + dp'. The LHS of Distribution function for MB and FD system (equation (2) and equation (6)) calculate the occupation index. Its value decreases when the statistical weight factor associated with a particular energy level is high. In a FD system, the condition $g_i \gg n_i$ is usually true. High value of statistical weight factor in comparison to number of particles in a FD system substantially reduces the probability that a quantum state is filled with more than one particle. This condition is also readily met at high temperature and low density. In that limit the quantum mechanical effect of degeneracy pressure is insignificant however the particles naturally maintain occupation index below unity and matches with the characteristics of ideal gas molecules in that limit.

2.3 Graphical Analysis

In Figure 1, distribution law for FD system is compared with a MB system by plotting distribution function (occupation index) against normalized energy.

The normalized energy is defined as

$$\varepsilon_n = \frac{\varepsilon-\mu}{KT} \quad (11)$$

The energy of the system is scaled with the thermal energy associated with the system of particles.

The graph is plotted at room temperature, T=300K. The choice of this temperature is relevant in the present context, where the idea is to observe the behaviour of electrons in classical limit (room temperature or above). It is observed that both the Distribution law (MB and FD) maintains an asymptotic behaviour after a certain value of energy. Thus, the probability to find a particle is zero when the energy of the level approaches infinity. From MB graph it is revealed that the maximum value of occupancy number at room temperature is approximately unity and it decreases exponentially with energy. The graph is observed at $\mu = KT$. When energy is appreciably larger than thermal energy, the distinction between two statistics eventually disappears. It happens at $\varepsilon_n \sim 3$ or above

The behaviour of FD statistics is also studied for relatively high value of chemical potential at T=300K. Figure 2 depicts the nature of Distribution function against normalized energy at $\mu = 2KT$. It is observed that the occupation index value is relatively high at low energy limit when chemical potential is greater than thermal energy. In addition to that, it is observed that the chemical potential can influence the critical value of normalized energy above which the quantum statistics reduces to classical.

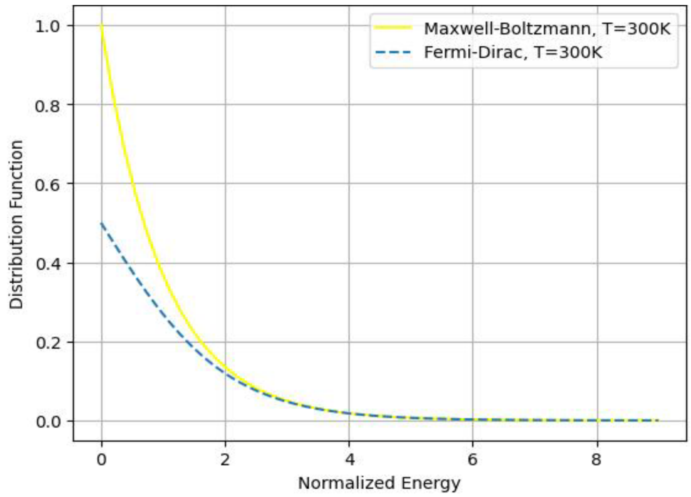


Figure 1: The figure shows the comparison of FD Distribution law with MB distribution law at room temperature (T=300 K) when chemical potential (μ=KT) is comparable to thermal energy.

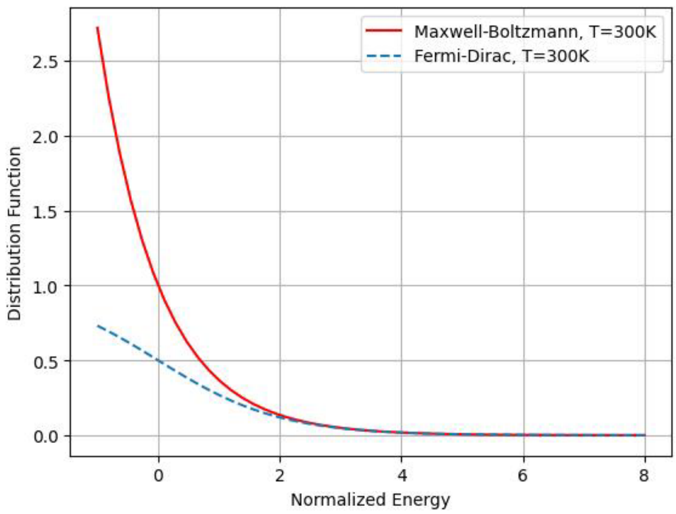


Figure 2: The figure shows the comparison of FD Distribution law with MB distribution law at room temperature (T=300 K) when chemical potential is twice the thermal energy (μ=2KT).

In Figure 1, the deviation of FD statistics from MB reduces less sharply in comparison to that in Figure 2. At room temperature, it is possible to observe the equivalence of FD and MB statistics as ε_n approaches 4. In the classical limit, the fermions have small value of distribution function or occupation index. The thermodynamics of electron gas in classical limit is similar to ideal gas molecules. The figures also establish the fact that there is negligible probability to find particle at an energy level much higher than the chemical potential of the system.

If the temperature of FD system ($T \ll T_F$) lies much below the Fermi temperature, TF, the Fermi gas behaves quantum mechanically. Under this condition, $\mu = \epsilon_F\left(1 - \frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right)$. Moreover, Fermi energy $\epsilon_F \propto n^{2/3}$. Thus the effect of temperature and density in governing the strength of distribution function (f_{FD}) can be estimated.

2.4 Electrons in laboratory Dusty Plasma (DP) set up

The laboratory dusty plasma set up is an ensemble of electrons, ions and dust that can exhibit collective behaviour. Dusty plasma is used as a model system to diagnose the physical properties of many body system at its atomistic level [6, 9].

In the laboratory dusty plasma systems the typical value of number density for electron species is $n_e \sim 10^{15} m^{-3}$ and spin degeneracy term for electron is 2 [11, 12]. The expression for Fermi energy is obtained considering a Fermi sphere with radius equal to Fermi momentum. Now, the Fermi energy is defined as- $\epsilon_F = \frac{p_F^2}{2m}$ where, $p_F = \left(\frac{3N}{8\pi V}\right)^{\frac{1}{3}} h$, where N is the total number of fermions and V is the spatial volume.

The Fermi energy of electrons in DP system can be calculated using the relation [2]-

$$\epsilon_F = \left(\frac{3n_e}{8\pi}\right)^{\frac{2}{3}} \frac{h^2}{2m} \tag{12}$$

Here, $n_e \sim 10^{15} m^{-3}$, $h = 6.636 \times 10^{-34} JS$, $m = 9.1 \times 10^{-31} kg$

Thus, ϵ_F=1.3×10^{-29}J. Now,

$$\epsilon_F = KT_F \tag{13}$$

The Fermi temperature corresponding to the calculated value of Fermi energy for dusty plasma electrons is given as $T_F = 10^{-6}$K. The Fermi temperature is much smaller than typical temperature of dusty plasma electrons ($T_e \sim 10^4$K). Hence, the electron gas is weakly degenerate and it obeys Maxwell Boltzmann distribution in typical laboratory conditions. Such theoretical considerations have yield experimentally verifiable structural and thermodynamic properties of many body systems [13, 14].

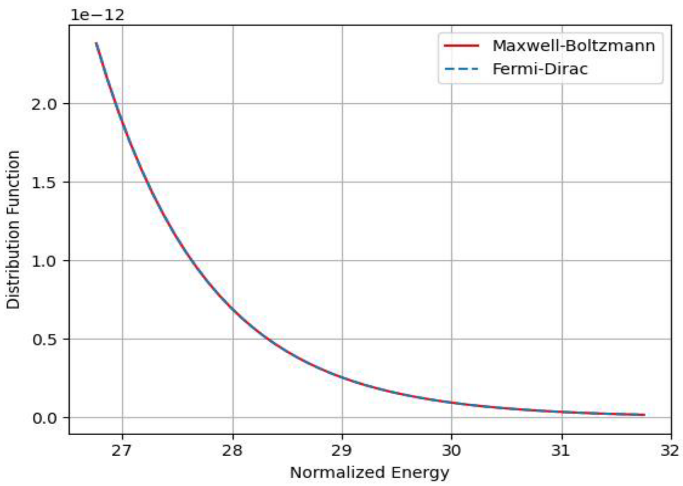


Figure 3: The figure shows the comparison of FD Distribution law with MB distribution law for dusty plasma parameters-$T_e \sim 10^4$K and $n_e = 10^{15}m^{-3}$. Figure depicts the nature of electron gas in typically laboratory dusty plasma parameters.

The plasma electrons in laboratory dusty plasma set up, obeys MB distribution for the entire range of normalized energy when the temperature $T_e \sim 10^4$K and density $n_e = 10^{14} - 10^{15}m^{-3}$. The Coulomb coupling parameter is defined as the ratio of potential to thermal energy of species in dusty plasma environment. It measures the degree of correlation between the particles [15].

$$\Gamma = \frac{e^2}{4\pi\varepsilon_0 r_{av} KT} \tag{14}$$

$$\Gamma = \frac{(1.6\times10^{-19})^2}{4\times3.14\times8.85\times10^{-12}\times10^{-5}\times1.3\times10^{-23}\times10^4} \tag{15}$$

$$\Gamma = 1.7 \times 10^{-4} \ll 1 \tag{16}$$

The above condition justifies the ideal gas approximation and establishes the fact that electrons can behave like ideal gas molecules in classical limit in typical laboratory condition. The chemical potential for electrons in laboratory dusty plasma set up may be defined as-

$$\mu = KT_e \ln\left(n_e \lambda_{T_e}^3\right) \tag{17}$$

Here, $T_e \sim 10^4$K, $n_e \sim 10^{15}m^{-3}$, the mean thermal wavelength λ_{T_e} may be calculated with dusty plasma parameters using equation (3). For the present case,

$$\mu \sim -36.92\times10^{-19} \tag{18}$$

Thus, the chemical potential for Fermi gas in dusty plasma is far less than unity. The result is true for typical laboratory condition. This is another important criteria that satisfies the consideration for MB distributed electrons in Dusty plasma.

3 Conclusions

In the present article, the behaviour and characteristics of FD statistics is studied and it is compared with Classical statistics. The condition for equivalence of two statistics is discussed. FD Distribution law in low temperature limit is also discussed. Thermodynamic parameters have a great role in tuning the occupation index. The reduction of FD statistics to MB statistics is justified in the context of laboratory dusty plasma electrons. The inequality relation (9) is obtained based on approximation and it might depend on thermodynamic parameters. The relation may guide for an approximate value of critical normalized energy beyond which the MB and FD statistics would become equivalent. In the future communication, the author may try to extend the work for a wider parameter space to explore the temperature and density range suitable for maxwellian electrons. Moreover, it will be possible to observe the impact of electron distribution on the thermodynamic properties of many particle systems such as dusty plasma.

The present work is useful to validate the presence of MB distributed electrons in dusty plasma. The analysis of FD statistics in Classical limit justifies the fact. It is well known that the condition $n_e \lambda_{T_e}^3 > 1$ leads to breakdown of the assumption that electrons behave classically. Usually at high enough densities or reasonably low temperature, the above mentioned condition may become true. Under such circumstances, the electrons no longer behave classically. Therefore, the Quantum mechanical effects turn out to be important for governing the dynamics of particles in such thermodynamic conditions.

The non maxwellian electrons are often encountered in space dusty plasmas. In non-thermal plasmas, the propagation of acoustic waves is explained with the consideration of electron distribution that deviates from Classical assumption. In such dusty plasmas, electrons are energetic enough and long tail distribution is accounted [16].

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