CIRCULAR MOTION IN A STATIC AND STATIONARY CHARGED ANTI-DE SITTER BLACK HOLE IN $f(R)$ GRAVITY

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Abstract: In this work, the geodesics of a test particle in a static and stationary charged anti-de Sitter black hole in a $f(R) = R - 2\alpha\sqrt{R - 8\Lambda}$ gravity is investigated. Inside the outer event horizon of the static and stationary charged anti-de-sitter black hole, there exists a stable and periodic circular orbit that is not coming out of the black hole but not falling into its singularity in the background of $f(R)$ gravity.

Keywords: Test particle; Black hole; Circular orbit; $f(R)$ gravity

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1 Introduction

One of the most convenient ways to investigate the geometrical structure around a black hole in the background of anti-de-Sitter spacetime, and to address certain issues such as dark energy, cosmological inflation is to study the geodesic equation of motion of very small and light particles (as compared to the black hole's size and mass) in and around the black hole. Subramanyan Chandrasekhar [1] was among the first to analyse two types of test particle orbits: orbits of the first kind and orbits of the second kind. Bound orbits of the third kind inside the outer horizon of the black hole were found by [2] for charged particles around the rotating charged black holes. Eva Hackman et al. [3] examined for neutral particles around rotating black holes.

Recently, a new type of charged spherically symmetric black hole solutions in the context of the $f(R) = R - 2\alpha\sqrt{R - 8\Lambda}$ gravitational scenario was discovered by Nashed and Capozziello [4] to address certain issues such as dark energy, cosmological inflation, or other unexplained gravitational phenomena. The geodesics and geodesic deviation for a stable circular orbit around the black hole has been discussed to understand the dynamics of spacetime in strong gravitational fields, to test theories of gravity, and to predict observable astrophysical phenomena. The presence of the term $\alpha\sqrt{R - 8\Lambda}$ modifies the gravitational dynamics at large scales. The term 8Λ represents a cosmological constant, linked to the energy density of the vacuum or dark energy. The modification by α, a constant, allows for a different curvature dependence than in standard general relativity, which explains cosmological observations such as the accelerated expansion of the universe. An asymptotically anti-de Sitter spacetime contains black-hole horizons only with a negative cosmological constant,$\Lambda < 0$ where the gravitational effects differ significantly from those of asymptotically flat black holes. These black holes are often studied in the context of theoretical physics, particularly in the study of holography and the AdS/CFT correspondence. The anti-de Sitter black hole is interesting because it provides a rich framework for understanding quantum gravity, especially through the AdS/CFT correspondence, which links gravity in an AdS space to a conformal field theory on its boundary[5]. This duality offers insights into strongly coupled quantum systems and has applications in areas like condensed matter physics, string theory, and high-energy physics, helping to bridge gaps between gravity and quantum field theory.

We examine geodesics of a test-charged massive particle in a stationary and static charged anti-de Sitter black hole and just outside its singularity in the $f(R)$ gravity where the tidal forces and radiation of gravitational waves of the black hole are assumed to be too weak to affect on the falling test particle when acted upon. We assume the generic properties of the black hole metric intact inside from the disturbance of the perturbative instabilities ([6],[7]).

This work is organized as follows: In the second section, we revisit the black hole solutions in the static and stationary charged anti-de Sitter black hole in a $f(R) = R - 2\alpha\sqrt{R - 8\Lambda}$ gravity. In the third section, the geodesic equation of motion of a test particle after the discussion of the black hole solution is obtained. In the fourth section, we discuss about the event horizons. In the fourth section, we examine the geodesic motion of the test particle. Finally, we present our conclusions of the work.

2 Black hole solutions

The model of gravity used to provide the black hole solutions is obtained by varying the action,

$$S = S_g + S_{EM}, \tag{1}$$

where S_{EM} is the electromagnetic field action and

$$S_g = \frac{1}{2\kappa} \int \sqrt{-g}(f(R) - \Lambda) \; . \tag{2}$$

S_g is the gravitational action with the cosmological constant Λ, the Ricci scalar R, the determinant of the metric g and the gravitational constant κ. In this study, S_{EM} represents the action of the nonlinear electrodynamics field, which takes the form

$$S_{EM} = -\frac{1}{2}F^{2s}, \tag{3}$$

where $s \geq 1$ is an arbitrary parameter that is equal to 1 for the standard Maxwell theory and $F^2 = F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu} = 2A_{\mu,\nu}$ with A_μ being the gauge potential 1-form, and the comma denotes the ordinary partial differentiation. The resulting Maxwell $f(R)$ field equations after the variation of the action with the metric $g_{\mu\nu}$ are

$$R_{\mu\nu}f'(R) - \frac{1}{2}g_{\mu\mu}f(R) - 2g_{\mu\nu}\Lambda + g_{\mu\nu}f'(R) - \nabla_\mu\nabla_\nu f'(R) = 8\pi T_{\mu\nu}, \tag{4}$$

$$\partial_\nu(\sqrt{g}F^{\mu\nu}) = 0 \; , \tag{5}$$

where $R_{\mu\nu}$ is the Ricci tensor, $F^{\mu\nu}$ is the electromagnetic field strength tensor, $f'(R) = \frac{df}{dR}$ and stress-energy tensor $T_{\mu\nu}$ is given below,

$$T_{\mu\nu} = \frac{1}{4\pi}[g_{\alpha\beta}F^\alpha_\mu F^\beta_\nu - \frac{1}{4}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}] \; . \tag{6}$$

The particular model considered to obtain the solutions is given by the function

$$f(R) = R - 2\alpha\sqrt{R - 8\Lambda} \; , \tag{7}$$

where α is a non-zero dimensional parameter that causes solutions to differ from the typical general relativity solutions. The value of α lies in between $0 < \alpha < 0.5$ to ensure the theoretical model stable and physically meaningful. Otherwise, the model leads to instabilities in the solutions.

3 Geodesic equations of motion

Geodesic equations for test particles of charge q and mass m in the Kerr-Newman metric were derived in [8]. Applying the Hamilton-Jacobi formalism, an orbital trajectory of a test-charged massive particle in a static and stationary charged anti-de Sitter black hole with a non-zero cosmological constant ($\Lambda < 0$) is described by the total energy of the particle E, the azimuthal component of the angular momentum L as observed by a stationary observer at infinity, and the Carter constant Q related with the total angular momentum of the particle in the $f(R) = R - 2\alpha\sqrt{R - 8\Lambda}$ gravity. The solution to the Maxwell field equation in the static and stationary charged anti-de Sitter black hole space-time with a non-zero cosmological constant ($\Lambda < 0$) in the Boyer-Lindquist coordinates (t, r, θ, ϕ) is given by the line element[4], (see also in [9]),

$$ds^2 = -g(r)dt^2 + g(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{8}$$

where

$$g(r) = \frac{12\alpha r^4 + 3\alpha r^2 y^2 - 2ry^2 + 2y^2}{6r^2y^2\alpha}, \tag{9}$$

with the scale parameter, $y^2 = -3/\Lambda$. The line element reduces to Reissner-Nordstrom black hole solution if $\Lambda \to 0$, and $\alpha \neq 0$ and the mass and charge of the black hole are realized respectively as $M = \frac{1}{6\alpha}$ and $q_h = \frac{1}{\sqrt{6\alpha}}$. The non-vanishing temporal component of the gauge potential is given by

$$A_t = \frac{1}{\sqrt{3\alpha r}}. \tag{10}$$

The set of equations of motion of a test massive charged particle in the anti-de-Sitter spacetime is obtained as [10]:

$$\begin{aligned} \frac{dr}{d\lambda} &= \pm\sqrt{V(r)}, && (11)\\ \frac{d\theta}{d\lambda} &= \pm\frac{\sqrt{V(\theta)}}{r^2}, && (12)\\ \frac{d\phi}{d\lambda} &= \frac{L}{r^2\sin^2\theta}, && (13)\\ \frac{dt}{d\lambda} &= \Big(E - \frac{q}{\sqrt{3\alpha r}}\Big)g^{-1}(r)\ , && (14) \end{aligned}$$

where $m\lambda = \tau$ is the proper time and

$$\begin{aligned} V(r) &= \Big(E - \frac{q}{\sqrt{3\alpha r}}\Big)^2 - g(r)\Big(m^2 + \frac{Q + L^2}{r^2}\Big)\ , && (15)\\ V(\theta) &= \Big(Q + L^2 - L^2\sin^{-2}\theta\Big)\ . && (16) \end{aligned}$$

Here, $V(r)$ and $V(\theta)$ represent the resulting potentials that describe the motion of the test particle in the $r-$ and $\theta-$ directions, respectively. The orbit of the particle at some distance r will completely confine to the equatorial plane if $Q = 0$ at $\theta = \frac{\pi}{2}$. The test particle executes a spherical orbit, which is a non-equatorial orbit at some radius in the θ-direction. The total angular momentum of the particle is given by $\sqrt{Q + L^2}$. The typical natures of the potential $V(r)$ versus the radial distance are shown in figure 1 for the various values of angular momentum, $L = 2, 3, 4$, taken at the values of the test charge, $q = 0.30$, $\alpha = 0.11, y = 2 \times 10^2, m = 0.2, Q = 6$.

4 Horizons

The metric has an intrinsic singularity from the horizon definition. The event horizons are given by the roots of $g(r) = 0$ i.e.

$$\begin{aligned} 12\alpha r^4 + 3\alpha r^2 y^2 - 2ry^2 + 2y^2 &= 0,\\ (r - r_+)(r - r_-)(r - r_c)(r - r_c^*) &= 0, \end{aligned}$$

where

$$r_\pm = \frac{-\mathcal{M} \pm \sqrt{Y_+}}{\mathcal{A}},\ \ r_c = \frac{\mathcal{M} \pm \sqrt{Y_-}}{\mathcal{A}} = r_c^*, \tag{17}$$

$$\begin{aligned} \mathcal{A} &= 2\sqrt{3\alpha},\ \ \mathcal{C} = \frac{\sqrt{\alpha}y^2}{4\sqrt{3}},\ \ \mathcal{D} = -\frac{y^2}{4\sqrt{3}\sqrt{\alpha}},\ \ \mathcal{E} = \frac{y^2}{\sqrt{3}\sqrt{\alpha}},\ \ \beta = \frac{\mathcal{C}^2 + \frac{\mathcal{A}\mathcal{E}}{3}}{\mathcal{A}^2},\ \ \mathcal{M} = -\sqrt{\mathcal{A}^2\Theta - \mathcal{A}\mathcal{C}},\\ \Theta &= \sqrt{\beta}\cosh\left[\frac{1}{3}\cosh^{-1}\left(\frac{\mathcal{C}^3 + \mathcal{A}\mathcal{D}^2 - \mathcal{A}\mathcal{C}\mathcal{E}}{\beta^{3/2}}\right)\right],\\ \mathcal{N} &= -\frac{\mathcal{A}\mathcal{D}}{\mathcal{M}},\ \ Y_\pm = -\mathcal{M}^2 - \mathcal{A}\Big(3\mathcal{C} \pm \mathcal{N}\Big). \end{aligned} \tag{18}$$

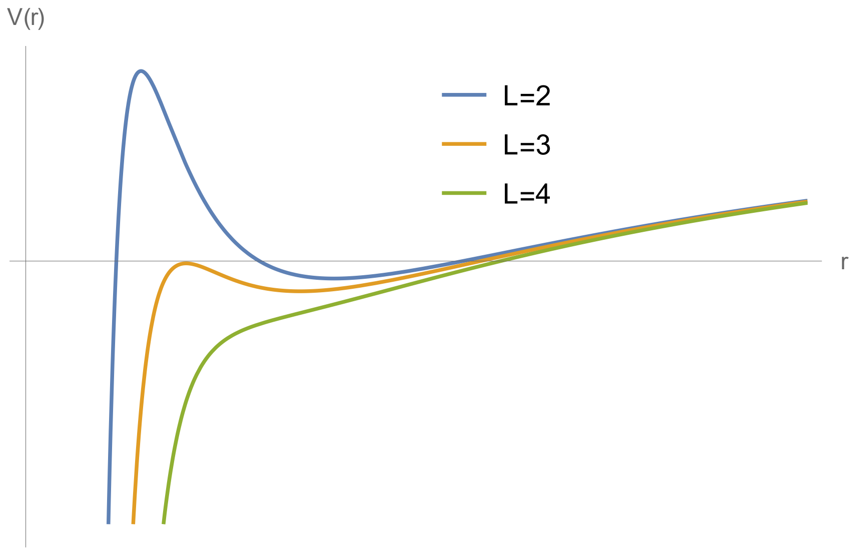

Figure 1: Variation of radial potential plot of a charged particle for the various values of L taken at the values of $q = .30, \ \ \alpha = 0.11, \ \ y = 2 \times 10^2, \ \ m = 0.2, \ \ Q = 6$.

Here, the two horizons: r_+, which represents the outer event horizon and $r_- < r_+$, which represents the inner event horizon of the black hole are obtained if $Y_+ > 0$ and the cosmological horizon r_c is obtained if $Y_- > 0$. A naked singularity may yield if $Y_+ < 0$ and an extremal black hole if $Y_+ = 0$. The possible locations of horizons are shown in figure 2.

5 Stable and periodic orbits inside the black hole

The circular orbit in the effective potentials $V(r)$ in the $r-$ direction is determined at some radius r, where the net force is zero by the conditions:

$$V(r) = 0, \ \frac{dV(r)}{dr} \equiv V'(r) = 0. \tag{19}$$

For small perturbation of the test particle from the stable orbit, the particle tends to return to the equilibrium position rather than moving further away or collapsing inward, which is understood by looking at the second derivative of the potential. For the orbit to be stable, any small perturbation (a slight displacement in radius) results in a restoring force that pulls the object back to the equilibrium orbit, which is given by the second derivative of the potential. If the second derivative is negative, this means that the potential is concave down, and the force will act in a way that restores the object to the equilibrium position after a small perturbation. This corresponds to a stable orbit. Thus, the stability condition for the circular orbits is given by

$$\frac{d^2V(r)}{dr^2} \equiv V''(r) > 0, \tag{20}$$

which ensures that the equilibrium is a local minimum of the potential, leading to a restoring force that stabilizes the orbit. The circular orbits happen completely in the black hole equatorial plane when $Q = 0$ at $\theta = \frac{\pi}{2}$ (see in [11]).

5.1 Circular orbits of a test charged massive particle inside the static and stationary charged de sitter black hole

Equatorial circular orbits with some radius r are given by the stability conditions of combined equations (19). Solving them for the energy, E and angular momentum, L^2 of test particles in the static and stationary charged de

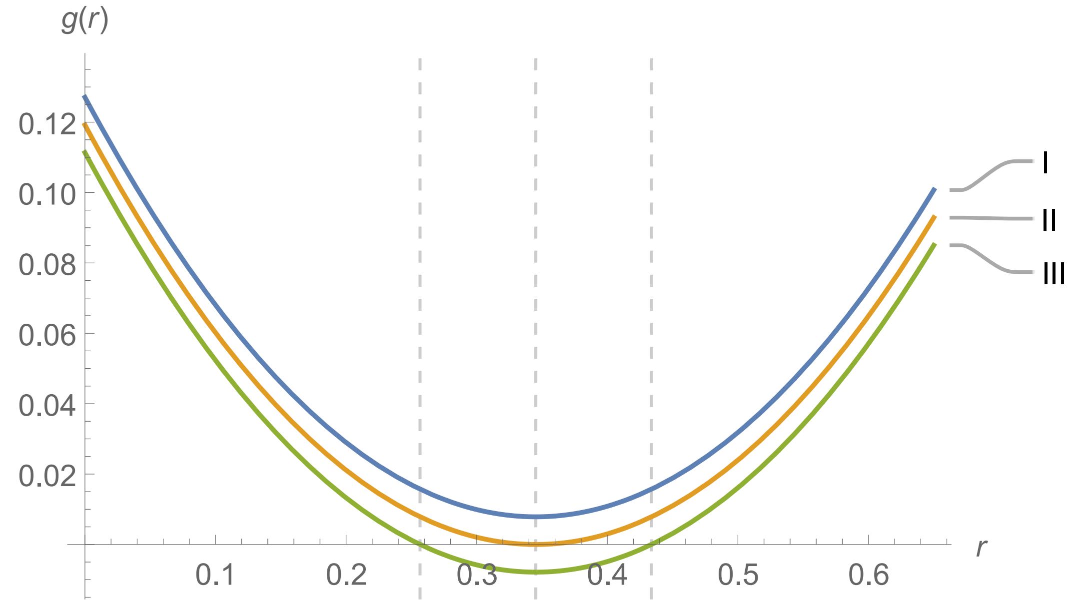

Figure 2: Schematic plot showing the event horizons of the black hole. The upper curve-I represents a naked singularity, the intermediate curve-II represents an extremal black hole and the lower curve-III represents a black hole with two event horizons.

Sitter spacetime, we obtain two pairs of solutions for E and L as

$$\begin{aligned} L_{\pm}^2 &= \frac{\Delta - \sqrt{q^2 r^4 \left(12\alpha r^4 + y^2\left(3\alpha r^2 - 2r + 2\right)\right)^2 \left(m^2\left(6\alpha r^2 - 6r + 8\right) + q^2\right)}}{y^2\left(3\alpha r^2 - 3r + 4\right)^2}, \qquad (21)\\ \text{where } \Delta &= m^2 r^2\left(3\alpha r^2 - 3r + 4\right)\left(12\alpha r^4 + (r-2)y^2\right) + 12\alpha q^2 r^6 + 3\alpha q^2 r^4 y^2 - 2q^2 r^3 y^2 + 2q^2 r^2 y^2.\\ E_{\pm} &= \frac{q\left(12\alpha r^4 + y^2\left(-3\alpha r^2 + 4r - 6\right)\right) \pm \left(12\alpha r^4 + y^2\left(3\alpha r^2 - 2r + 2\right)\right)\sqrt{m^2\left(6\alpha r^2 - 6r + 8\right) + q^2}}{2\sqrt{3}\sqrt{\alpha} r y^2\left(3\alpha r^2 - 3r + 4\right)}. \end{aligned}$$

The real conditions are satisfied if and only if $(3\alpha r^2 - 3r + 4) > 0$. The plots for various values of the cosmological constants and rotating parameters are shown in figures 3, 4. The stability condition $\frac{d^2V(r)}{dr^2} > 0$ for circular orbits inside the outer horizon $0 < r < r_+$ is satisfied for the pair of solution (E_+, L_+) and (E_-, L_-) for all $\alpha < 3/16$. The typical plot showing the stable circular orbit inside the outer event horizon of the static and stationary charged black hole for the asymptotically flat solution is shown in figure 5.

The proper period of the circular orbit at some value of r is given by

$$T_\lambda = \frac{2\pi r^2}{L_\pm}\,, \qquad (22)$$

and the coordinate period is given by

$$T_t = \frac{2\pi r^2}{L_\pm}\frac{\left(E_\pm - q/\sqrt{\alpha}3r\right)}{g(r)}\,. \qquad (23)$$

5.2 Circular orbits of a test neutral massive particle inside the stationary and static charged black hole

Inside the outer event horizon of the static and stationary charged black hole, stable and periodic circular orbits exist for neutral massive particles. We find two pair of solutions for E and L with some radius r as

$$L_\pm(r) = \pm\frac{mr}{y}\sqrt{\frac{ry^2 - 2y^2 + 12\alpha r^4}{4 - 3r + 3r^2\alpha}}\,, \qquad (24)$$

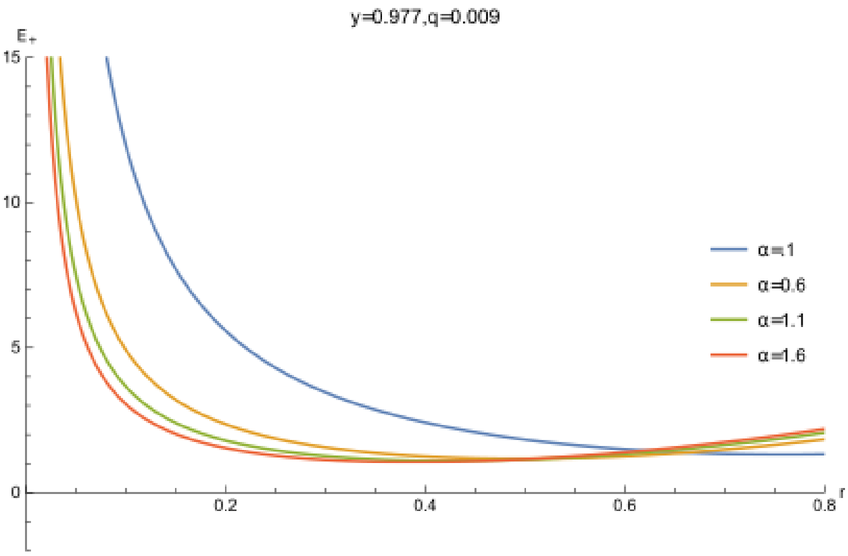

Figure 3: The plot shows the variation of E_+ with the radial distance from the black hole at different values of α, at $y = 0.977, q = 0.009$.

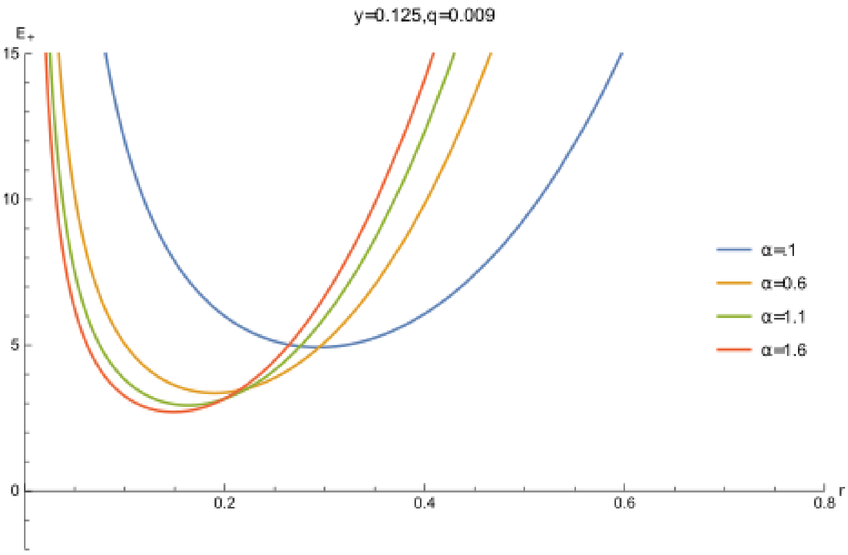

Figure 4: Variation of E_+ with the radial distance from the black hole at different values of α, at $y = 0.125, q = 0.009$.

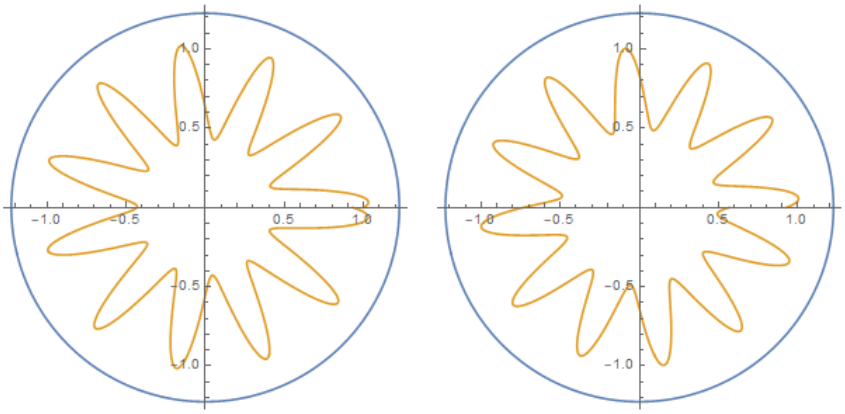

Figure 5: Schematic plot showing the stable circular orbits for test charged massive particle

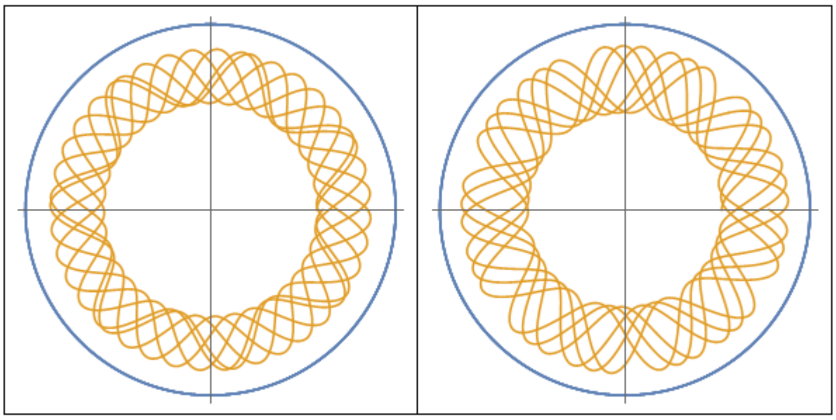

Figure 6: Schematic plot showing the stable circular orbits for neutral massive particle inside the inner event horizons

$$E_{\pm}(r) = \frac{\pm m}{ry^2\sqrt{\alpha 6}} \frac{12r^4\alpha + 2y^2 - 2y^2r + 3y^2r^2\alpha}{\sqrt{4 - 3r + 3r^2\alpha}} \; . \tag{25}$$

The stability condition $\frac{d^2V(r)}{dr^2} > 0$ for circular orbits inside the inner horizon $0 < r < r_+$ is satisfied for the pair of solution (E_+, L_+) and (E_-, L_-) for all $m = 1$, with the condition that $\alpha > \frac{3r-4}{3r^2}$.The typical plot showing the stable orbit inside the static and stationary charged black hole is depicted in figure (6).

5.3 Spherical orbits of a neutral particle inside the static and stationary charged black hole

Using the equation (19) we obtain two pairs of solutions for E and azimuthal impact parameter $b = \frac{L}{E}$ of a test massive charged particle:

$$E_{\pm} = \pm\frac{m}{ry^2\sqrt{6\alpha}} \frac{(12r^4\alpha + 2y^2 - 2y^2r + 3r^2y^2\alpha)}{\sqrt{4 - 3r + 3r^2\alpha}} \; , \tag{26}$$

$$b_{\pm} = \pm\frac{ry\sqrt{6\alpha}}{m} \frac{\sqrt{m^2r^2(ry^2 + 12r^4\alpha - 2y^2) - Qy^2(4 - 3r + 3r^4\alpha)}}{(12r^4\alpha + 2y^2 - 2y^2r + 3r^2y^2\alpha)} \; .$$

It is verified that for $r \neq 0, y \neq 0$ and $\alpha \neq 0$, stable orbits are realized for the pair of solution(E_+, b_+) with $0 < Q < Q_{max}$, where Q_{max} is a root of the marginal stability equation of $V_r'' = 0$ with the condition that $\alpha > \frac{2(ry^2 - y^2)}{3r^2(4r^2 + y^2)}$.

6 Discussion

In this work, we have obtained the geodesic equations that describe the motion of a test particle in a static and stationary charged anti-de Sitter black hole in $f(R)$ gravity, specifically to examine how a free particle moves under the influence of spacetime curvature by means of the Hamiltonian-Jacobi equation of motion. The particle's motion is influenced by both the gravitational and electromagnetic fields of the black hole, with the presence of $f(R)$ gravity altering the overall curvature and geometry of the spacetime. The AdS background adds constraints on the particle's motion due to the nature of the spacetime's boundary conditions. A circular orbit of a test charged massive particle around the static and stationary charged de Sitter black hole is obtained for all conditions that $\alpha < 3/16$ and $(3\alpha r^2 - 3r + 4) > 0$. A spherically stable orbit in the internal space-time domain $0 < r < r_-$, where r_- is the black hole's inner horizon is, are found with the condition that for $r \neq 0, y \neq 0$, $\alpha \neq 0$, and $\alpha > \frac{2\left(ry^2 - y^2\right)}{3r^2(4r^2+y^2)}$. Lastly, it is found that the circular orbit, as a relativistic limit of the test particle's spherical orbit, exists in the stable and bound orbits in the black hole in the framework of the anti-de Sitter spacetime.

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